The Alternating Series Theorem with Examples

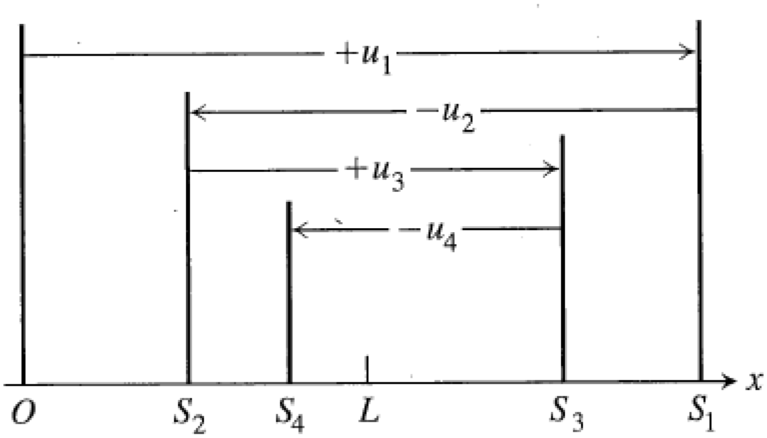
Now we look at series in which the **terms alternate in sign**. In order to fit this category of series the signs of the series must alternate (even terms are positive while odd terms are negative or vice versa), not just be a mixture of positive and negative terms. Almost all of them have a $(-1)^{n+1}$ as part of the formula for a_n (or it could be $(-1)^n$). The exception to that last statement would be a trig function whose values alternate in sign, so the -1 is not needed in the expression for a_n.

In order to prove that an alternating series is convergent, we use the **Alternating Series Theorem** (also known as Leibniz's Theorem).

Alternating Series Theorem
The series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ and $\sum_{n=1}^{\infty}(-1)^{n}a_n$ will converge if **both** of the conditions below are met:
1) $\lim_{n\to\infty} a_n = 0$ (the terms approach zero), and b) $a_{n+1} \le a_n$ for all n (the terms decrease in size)

Notice that the first condition is actually the n^{th} term test. If a series fails this first condition, then our reason for saying it is divergent is that the series did not pass the n^{th} term test.

This diagram illustrates what is happening when you add the terms of an alternating series:



The terms are getting smaller and approaching zero. Thus, the sum will "zoom in" on a Limit, which will be the sum of the series. Because we will add one term, subtract the next, then add a smaller number, and subtract another that is even smaller, the sum of the series will approach some "middle number" that will be our finite limit.

This theorem is relatively easy to use; you just have to make sure that you have terms with alternating signs and that you meet the two conditions.

To review:

If we have an alternating series (one whose terms alternate in sign), it will converge if it meets these two conditions:

1 - $a_{n+1} \leq a_n$ for all n (the terms decrease in size), and

2 - $\lim_{n\to\infty} a_n = 0$ (the terms approach zero)

****** if a test fails condition #1, then it has failed the n^{th} term test.

******* As you do these problems, you may abbreviate the Alternating Series Test as the AST.

Here are some examples.

A: Determine if the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converges or diverges.

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$	This is our series.
$\lim_{n\to\infty} \frac{1}{n} = 0$ The terms approach zero	First I check to see if the limit of a_n is zero.
$\frac{1}{n+1} < \frac{1}{n}$ for all n > 1 The terms are decreasing	Now we show that the terms are decreasing in size. It was easy to do with this series.
Thus, by the alternating series theorem the series converges.	This series meets all the criteria, so it converges.

Cool eh?

**** a special note** about this series that we just proved convergent. Notice that it I take out the -1 factor that makes the terms alternate in sign, I have the Harmonic Series (which we already know is divergent). This series is also very common and known throughout the mathematical world as the **convergent Alternating Harmonic series**. When you see it, you do not have to go through the proof, just state that it is the convergent Alternating Harmonic Series.

This next example is more complicated. In this one, it is not self evident that the terms are decreasing and we will have to include more proof. Ugh!

B: Determine if the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$ converges or diverges.

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$	This is our series.
$\lim_{n\to\infty}\frac{\ln n}{n}=\lim_{n\to\infty}\frac{\frac{1}{n}}{1}=0$ The terms approach zero	First I test to see if the terms are approaching zero. I used L'Hopital's rule.
$\frac{\ln(n+1)}{n+1}\overset{?}{<}\frac{\ln n}{n}$ n ln (n + 1) $\overset{?}{<}$ (n + 1) ln n $\frac{d}{dn}\left(\frac{\ln x}{x}\right)=\frac{1-\ln x}{x^2}<0$ for all x > 1 Thus, from above, $a_{n+1} < a_n$ for all n > 1 The terms are decreasing.	Now I have to show that the terms are decreasing in size. It is not readily apparent here; both the numerator and denominator are different. Sometimes you can, since n > 1, multiply through and show that the inequality is true. However in this case, it is not really working well. Another technique is to **check the derivative of the a_n expression.** If you can show the derivative is always negative for n > 1, then you know the terms are decreasing.
By the AST, the series converges.	This series meets all the criteria, so it converges.

I expect you to always show conclusively that the terms are decreasing in size! Most of the time it is obvious, but sometimes it is not and more work will have to be done. Don't forget about the trick of using the derivative!

Let's look at some more examples.

C: Determine if the series $\sum_{n=1}^{\infty}(-1)^{n+1}\ln\left(1+\frac{1}{n}\right)$ converges or diverges.

$\sum_{n=1}^{\infty}(-1)^{n+1}\ln\left(1+\frac{1}{n}\right)$	This is our series.
$\lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)=\ln\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\right)=\ln 1=0$	First show that the terms approach zero. I always do this first, since if we do not get zero, we know that the series diverges and we can stop.
$\ln\left(1+\frac{1}{n+1}\right)<\ln\left(1+\frac{1}{n}\right)$	Now show that the terms are decreasing in size.
The series converges by the Alternating Series Theorem.	State your conclusion.

**** This last example shows how to end a proof when we can show that the terms are not approaching zero!**

D: Determine if the series $\sum_{n=1}^{\infty}(-1)^{n}\frac{3\sqrt{n+1}}{\sqrt{n}+1}$ converges or diverges.

$\sum_{n=1}^{\infty}(-1)^{n}\frac{3\sqrt{n+1}}{\sqrt{n}+1}$	This is our series.
$\lim_{x\to\infty}\frac{3\sqrt{n+1}}{\sqrt{n}+1}=3$ The terms do not approach zero so the series diverges by the n^{th} term test.	Find the limit of the n^{th} term. The limit is not equal to 0, so the series diverges. **Notice: The reason we give is that it has failed the n^{th} term test, not that it diverges because of the Alternating Series Test! The n^{th} term test is actually the first part of the AST.**